A SIMPLE MATHEMATICAL STUDY OF ANTERIOR DENTAL RELATIONS: PART I

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A mathematical study of the anterior dental relations could provide a rationale for the work of Neff and Bolton. Bolton used normal occlusions of permanent teeth to obtain his range of comparisons of tooth width ratios; a model that needs no normals is described in this study. Use of this model and its development could support or disprove further hypotheses regarding anterior occlusion. Manipulation of the model could allow a means of estimating the influence of several dental measurement variables on the relation between upper and lower dental arches.
The formulae allow changes in the form and size of one or both arches individually and a comparison of an upper arch to different upper arches and a lower arch to different lower arches; and further, they allow these combinations of upper and lower arches to relate to each other. A reference plane between the arches is established to relate opposing arches to each other. The model is developed using the archform and angular measures of Bonwill and the mesiodistal widths of Bolton. Scope is provided in the model to allow for alteration of antero-posterior relations, spacing and crowding. Values for different amounts of overjet of incisors and maxillary canine tip could be generated by altering the figures to those of other authors' hypotheticals or by the use of individual patient data.

Key Words: Mathematics, Archform, Overjet, Tooth width.

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Aim

The aim of this paper is to describe a mathematical model which provides better insights into anterior dental relations. A subset of aims is to develop the logical basis of the Bolton[1,2] analysis and more recent refinements by Ho[3].

The creation of a spreadsheet allows the study of hypothetical alteration of various dental measures and how this may change other measures. A reference plane is described which enables calculated comparisons between the two arches. From the mathematical model explained in this paper it is possible to quantify antero-posterior and lateral interarch relations.

Introduction

For dental occlusion, two dentate arches are needed. The anterior dental relations may be measured by the extent of the overbite, overjet, incisal angles and archform. Angle[4] quotes Hawley's[5] description of Bonwill's[6,7] model: "In this way two lines of occlusion are inferred, one for each arch, which may or may not have direct relation to the other. In reality it has been vague and indefinite. So far as the author is aware, none has comprehended its full meaning and importance." There is still a need for analytical tools with which anterior dental relations may be examined and quantified and, because of this, a rational development of concepts is impeded.

Neff's and Bolton's examination of interarch relations used tooth widths from normal dentitions and from these developed an ideal ratio; and also ranges within which overbite could be expected to be normal (Neff[8] and Bolton[1,2]). Bolton[1] makes the orthodontist reliant on his definition of normal because he (Bolton) has one defined overbite goal which dictates predetermined attributes of the dentition and occlusion. By use of this simple mathematical model it is possible, with alteration of factors such as tooth widths, archform and various intra-arch and interarch dental measurement variables, to comprehend and predict changes to overbite and overjet.

The mathematical model created for this study uses two arcs to represent the maxillary and mandibular arches meeting to represent anterior occlusion. Each of the arcs has a definable posterior line and are then related to each other as parallel lines, thereby creating a plane (Figure 1).

Materials and Method

For this study the following three assumptions are made:

1. A "line of occlusion" is used as an assumed concept to allow a curvilinear analysis. The line described would follow the incisal edges mesiodistally along the line of the teeth and along the ridge of the canines also in a mesiodistal path. This line appears to conform to the illustrations of Bonwill[6,7] and Hawley[5] and this assumption further requires that interproximal contact be very close to the widest mesiodistal dental measurement of any particular tooth. In contrast, Andrews[9] seeks to find an optimal orthodontic bracket position and uses the labial dental surface for his linear reference. The author's model does not use Andrews' line.
2. The anterior form of the dental arch has been described in various ways. In the author's model the arc of a circle, as first described by Bonwill[5,6] and illustrated by Tweed[10], is used.
3. In the "normal" arrangement of occlusion described by Angle[4], the maxillary canine tip lies between the

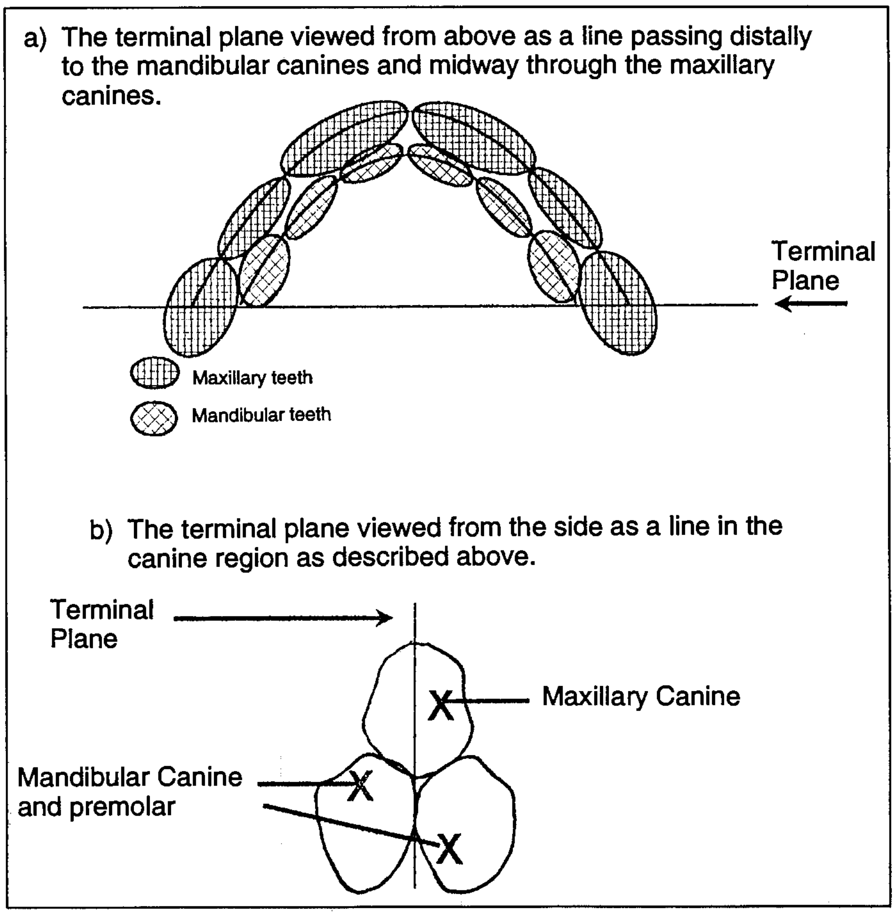

Figure 1. Diagrammatic representation of the position of the terminal reference plane.

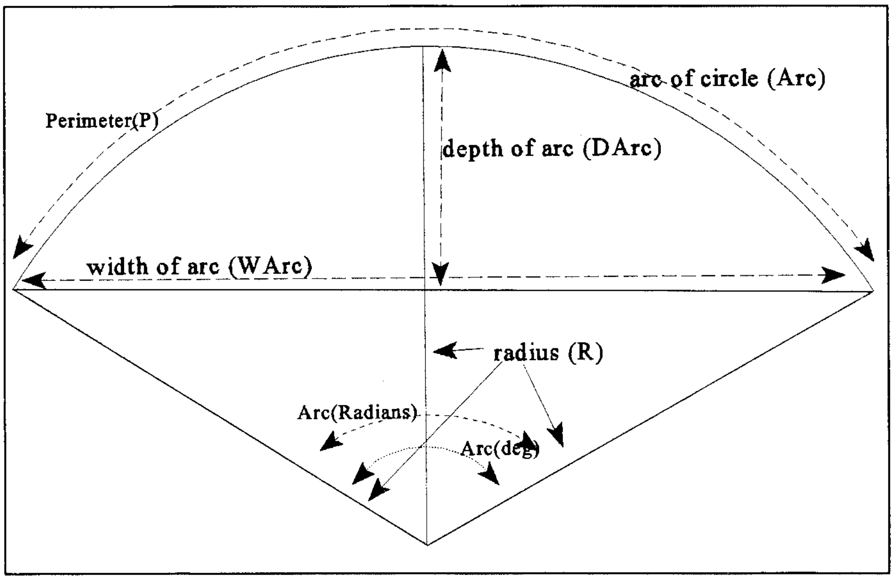

Figure 2. Diagrammatic representation of depth and width of arc.

mandibular canine and mandibular premolar. The posterior correlation between the maxillary and mandibular anterior segments will be the most distal point of the mandibular canine and a point calculated as half the width of the maxillary canine (Figure 1). This point does not place the maxillary canine between the mandibular canine and the first premolar when measured perpendicular to the arch at that point but this can be easily calculated in a more sophisticated model. Simple adjustments are available within this model. The distal two points for each arch describe a line. The two lines described are considered parrallel allowing them to both lie in a plane referred to as the reference plane. (The lines would possess corresponding X and Y co-ordinates and may differ in the Z co-ordinate. The Y co-ordinates between the line may be adjusted to account for factors such as canine position correction and Class II and Class III buccal relations.)

Terminology

The following terms are used in the text and some definitions are illustrated in Figure 2:

- **Arc** The line described in the preceding text. For this paper the line is a segment of a circle.
- **Arc (deg)** Angular measurement of the segment at the centre of a circle expressed in degrees (see Figure 2).
- **ARC** The same as Arc (degree) with the measure expressed in radians (see Figure 2).
- **R** Radius: the distance from the centre of a circle to any point on its circumference.
- **P** Perimeter: the length of the line as it describes the arc from one end to the other. This is the length of the arc: not the perimeter length of the complete circle (see Figure 2).
- **PMx** Maxillary perimeter.
- **PMd** Mandibular perimeter.
- **π** Pi: the mathematical constant.
- **DArc** Depth of the segment of the arc: the distance measured from the midpoint of a straight line joining the ends of the arc to the half way point on the perimeter.
- **WArc** Width of the segment of the arc: the linear (straight line) distance between the ends of the arc.
- **Sin** Sine: the trigonometric function.
- **Cos** Cosine: the trigonometric function.
- ***** A multiplication sign in a formula.
- **Tip** Angle of the long axis of the tooth crown to an occluso-gingival line perpendicular to the line of occlusion (perpendicular to the radius and tangent to the arc described as the line of occlusion).

DESCRIPTION	FORMULA	Bolton's figures	Example
Maxillary right central incisor width		8.7	8.7
Maxillary left central incisor width		8.7	8.7
Maxillary right lateral incisor width		6.7	6.7
Maxillary left lateral incisor width		6.7	6.7
Maxillary right canine width		7.7	7.7
Maxillary left canine width		7.7	7.7
Maxillary spacing or planned augmentation +ve crowding or planned reduction -ve			5
SUM Maxillary anteriors (Pmx)	cents + lats +canines/2 + Mx Space	38.5	43.5
Mandibular right central incisor width		5.2	5.2
Mandibular left central incisor width		5.2	5.2
Mandibular right lateral incisor width		5.8	5.8
Mandibular left lateral incisor width		5.8	5.8
Mandibular right canine width		6.7	6.7
Mandibular left canine width		6.7	6.7
Mandibular (Md Space) spacing or planned augmentation +ve crowding or planned reduction -ve			-4
SUM Mandibular anteriors (PMd)	cents+lats+ canines + Md Space	35.4	31.4

Table 1. Calculation of the anterior perimeter of the Maxillary and Mandibular arches.

Formulae

Most computer spreadsheet programs use radians as the angular measure; this is calculated to the formula of :

Formula #1

$ARC=Arc(deg)*(\pi/180)$

the radius of each arc is calculated by rearranging the formula.

Formula #2

$P=2\pi*R*(1/2\pi/ARC)$.

Formula #3

therefore, $R=(P/2\pi)*(2\pi/ARC)=P/ARC$.

the depth of the arc is calculated by subtracting the length of the radius projecting past a line joining the two ends of the arc.

Formula #4

DArc=R*(1-Cos(ARC/2))

the transverse dimension is calculated by:

Formula #5

WArc = R*2*Sin(ARC/2).

Formula #6

Canine Overjet=

((WArc Mx-WArc Md)/2).

Formula #7

Canine Adjustment=

Lateral Overjet*Tan(90-arc÷2)degrees.

These assumptions and formulae provide a starting point for analysis. It is possible to substitute parabolic, ellipsoid or other curvi-linear equations for comparison; indeed, three-dimensional curves would allow an even greater scope for analysis. The premise for the canine relations may be modified and even factored into the equation also allowing for Class II and Class III buccal relationships; and, in the case of a missing lateral, for example, to use the canine as a lateral and a premolar as a canine.

A spreadsheet was constructed using "Lotus Improv 2.0" with an IBM compatible computer with an Intel 80486DX33 chip.

Constructing the model

Measurements are tabulated using Table 1 (similar to Bolton's[1,2] analysis). The mesio-distal tooth measurements of the maxillary and mandibular incisors and canines are entered. For the maxillary arch, the total width of the canines is halved and added to the incisor widths. For the mandibular arch, the total of the incisors and both canines is calculated.

To allow for crowding and spacing, further calculations are required. Table 2 shows an additional row with a positive sign for spacing and augmentation and a negative sign for crowding and stripping. The resulting figure is the perimeter length (P) used in calculating "DArc" and "WArc".

Computer spreadsheets enable a formula to relate cells within the spreadsheet. Such factors as perimeter length changes, however, which are due to the labio-lingual width of the tooth in the contact point region, may create a perimeter value slightly greater in both arches when the tooth width is calculated with the angle of the arc (for example, the arc described below may be divided by the number of contact points, multiplied by tooth thickness and the appropriate trigonometric function).

Arc calculations

The depth and width of the arcs of each arch are calculated individually using their respective perimeter lengths. First, the angle of the arc is nominated and is converted from degrees to radians. The conversion of degrees into radians uses Formula #1.

The radius of the arc is required for determining other dimensions of the segment. The radius is calculated by using Formula #3. The most distal reference point is the distal of the mandibular canines and half the width of the maxillary canines. The depth of the segment must be calculated using Formula #4. To calculate the transverse position of the maxillary canine cusp tip and the distal of mandibular canine, the width of the segments are required (Formula #5).

Arc calculations are presented in Table 2; for the sake of brevity, the perimeter is taken from the calculation of Bolton figures in Table 1 and the arc is 120 degrees. The figure of 120 degrees is, by deduction, that used by Bonwill[6,7] Hawley[5] and Tweed;[10] they use the radius to describe a length along the circumference which produces an equilateral triangle for the right and left sides by adding the angles at the centre of the circle (60+60 degrees) subtended by the radii defining the limits of the arc.

Relating maxillary and mandibular segments

The mandibular segment depth is subtracted from the maxillary segment depth to provide an overjet figure (DArc,Mx-DArc,Md). To calaculate the incisor overjet, the mandibular depth is subtracted from the maxillary depth; any

Description	Formula	Maxilla(mm.) (Bolton)	Mandible(mm.) (Bolton)
Perimeter(P)	(from Table 2)	38.5	35.4
Arc(Deg)		120	120
ARC	Formula #1	2.09	2.09
Radius(R)	Formula #3	18.38	16.90
Depth of Segment (DArc)	Formula #4	9.19	8.45
Width of segment (WArc)	Formula #5	31.84	29.28
Unilateral width of segment	WArc/2	15.92	14.64

Table 2. Calculation of the radius, depth and width of a segment given the angle of the arc of a circle and the length of the perimeter contained within that arc.

DESCRIPTION	FORMULA	EXAMPLE(mm)
Depth of maxillary segment	(from Table 3)DArc Mx	9.19
Depth of mandibular segment	(from Table 3)DArc Md	8.45
Maxillary depth minus mandibular depth	(Depth maxillary segment)-(Depth mandibular segment)	0.74
Class II trend (in mm.) positive Class III trend (in mm.) negative		0
Incisor Overjet	(Maxillary depth- mandibular depth)+(Class II/III trend)	0.74
Unilateral width of maxillary segment	(from Table 3)	15.92
Unilateral width of mandibular segment	(from Table 3)	14.64
Lateral overjet at the tip of the maxillary canine	(Unilat. width of max. seg.)-(unilat. width of mand. seg.)	1.28

Table 3. Relating the maxillary and mandibular segments to each other.

Class II discrepancy is added; and a Class III discrepancy is subtracted. The resulting figure is an estimate of incisor overjet.

The lateral overjet at the maxillary canine tip may be derived by subtracting the unilateral mandibular width from the corresponding maxillary measure (Formula #6) resulting in an overjet measure for the canines rather than a figure for the bilateral difference (see Table 3).

The A-P relations between the maxillary canine midpoint and the distal of the mandibular canine may require adjustment for a discrepancy of their distal reference lines due to the angle of the arcs. The wider of the arcs (DArc) has its most distal point carried anteriorly along the line of the radius (the angle of the arc has both an anterior and lateral vector). Using the example of Table 3, and assuming the maxillary canine is on the same line as the radius, an appropriate formula could be: Formula #7. In this example the change is 0.740mm which projects the maxillary arc forward, thereby increasing the incisor overjet to 1.48mm.

Discussion

This model is not sufficiently developed to be used as a clinical tool; instead, it is an aid to the visualisation of changes to antero-posterior and transverse overjet in light of a variation in dental measurements such as tooth width, arch form and inter-arch perimeters. There is scope for the inclusion of refining formulae to determine more accurately the arch perimeter and maxillary canine tip position - if these are required. The space that each tooth occupies mesiodistally may also be influenced by altering crown tip; it is possible to modify the spreadsheet to accommodate this measure. If more detailed measurements of each tooth are made it may be best to include them in a table such as Table 1 with each tooth's width, thickness, height and tip noted to enable calculation of the arc perimeter. A common reference plane is defined which allows interarch relations to be assessed.

Rather than using a fixed ratio or a range of ratios to guide prediction of anterior occlusion, this model provides a means of estimating the effect changes of various measures have on each other and makes it possible to forecast the consequences of crowding or spacing in an untreated or treated case.

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